CURVATURE OF A ZERO-POTENTIAL FORMING ELECTRODE

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Local characteristics such as the angle θ between the beam boundary and the zero equipotential or the angle at which the particles leave the emitter, both for the case of space-charge-limited emission, have been studied in [1, 2]. It is found that the particles start along the normal, while $\theta = 3\pi/8$. The present communication deals with the relation between the curvatures of the emitting surface, the particle trajectory, and a forming electrode at zero potential. The shape of the latter is considered for planar and axially symmetric beams in the absence of a magnetic field. Previous results [3, 4] are used for the solution of the equations for a regular beam, which is represented in the form of a series in x^1, with emission occurring from the surface with $x^1 = x_0^1 = \mathrm{const}$.

We have to derive the equation for the path near the starting point O with an accuracy sufficient to give correct values for the curvature and the derivative of the curvature at that point. The method of [5] is used for the analytic continuation of the potential given (with its normal derivative) on the trajectory taken as the beam boundary into the charge-free region. The equation for the equipotential that is correct near O allows us to establish the desired relation.

Let $x^1 = x^1(z, R)$, $x^2 = x^2(z, R)$ be an orthogonal coordinate system in the (z, R) meridional plane having g_{ik} as metric tensor, while $x^1 = x_0^1$ defines the emitting surface, where the current density $J = J(x^2)$ is assumed to be given. Passing to the physical components v_{x^1}, v_{x^2} of the velocity and to the arc lengths S and P of the curvilinear axes x^1 and x^2, we have

$$\frac{dS}{dt} = \frac{v_1}{\sqrt{g_{11}}} = \left(\frac{9J}{2}\right)^{1/3} S^{2/3}(1 + \Omega S), \qquad \Omega = \frac{4}{15} T,$$

$$T = \varkappa_1 + \varkappa_2, \qquad \frac{dP}{dt} = \frac{v_2}{\sqrt{g_{22}}} = \left(\frac{9J}{2}\right)^{1/3} S^{5/3}(\Lambda + \Theta S),$$

$$\Lambda = \frac{1}{5}\frac{J_P'}{J} - k_1, \quad \Theta = \frac{1}{10} T_P' - \frac{1}{2} k_{1S}' - \frac{23}{30}\varkappa_1 k_1 -$$

$$- \frac{4}{15}\varkappa_2 k_1 + \frac{7}{30}\varkappa_1 \frac{J_P'}{J} + \frac{1}{30}\varkappa_2\frac{J_P'}{J}. \tag{1}$$

Here $\varkappa_1$, $\varkappa_2$ and k_1, k_2 are the principal curvatures of the surfaces $x^1 = \mathrm{const}$, $x^2 = \mathrm{const}$ as calculated at $x^1 = x_0^1$; they are therefore functions of x^2.

From (1), we find the differential equation of the trajectory

$$\left(\frac{g_{11}}{g_{22}}\right)^{1/2} \frac{dx^1}{dx^2} = \frac{1}{S}\left[\frac{1}{\Lambda} + \left(\frac{\Omega}{\Lambda} - \frac{\Theta}{\Lambda^2}\right) S\right], \tag{2}$$

whose solution is defined by the expression

$$x^2 - x_0^2 = \frac{a_0 \Lambda}{2 b_0^{1/2}}(x^1 - x_0^1)^2 + \frac{a_0^{3/2}\Lambda}{b_0^{1/2}}\left[\frac{1}{4}\frac{a_1}{a_0^{3/2}} + \right.$$

$$\left. + \frac{1}{3}\varkappa_1 - \frac{1}{3}\left(\Omega - \frac{\Theta}{\Lambda}\right)\right](x^1 - x_0^1)^3, \tag{3}$$

in which a_k and b_k are the coefficients in the expansion of the elements of the metric tensor with respect to $(x^1 - x_0^1)$, it being understood that the values at O are used for all quantities in (3). In the (z, R) plane we introduce the local Cartesian coordinates X and Y linked to the emitting surface, with X directed along the normal and Y directed along the tangent at O:

$$X = (z - z_0)\cos\vartheta + (R - R_0)\sin\vartheta,$$

$$\cos\vartheta = \sqrt{g_{11}}\,\partial x^1/\partial z = \sqrt{g_{22}}\,\partial x^2/\partial R,$$

$$Y = -(z - z_0)\sin\vartheta + (R - R_0)\cos\vartheta,$$

$$\sin\vartheta = \sqrt{g_{11}}\,\partial x^1/\partial R = -\sqrt{g_{22}}\,\partial x^2/\partial z. \tag{4}$$

Note that ϑ is the angle between the normal to the emitter at O and the axis of rotation z.

We now need expansions of the functions $x^1 - x_0^1$, $x^2 - x_0^2$ with respect to X and Y. It is readily seen that

$$s_0 = a_0^{1/2}(x^1 - x_0^1) = X + k_1 XY - \frac{1}{4}\frac{a_1}{a_0^{3/2}} X^2 - \frac{\varkappa_1}{2} Y^2 +$$

$$+ \left[\frac{1}{6}\left(\frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2}\right) - \frac{1}{3}k_1^2\right] X^3 -$$

$$- \left(\frac{1}{6}\varkappa_{1P}' + \frac{1}{2}\varkappa_1 k_1\right) Y^3 + \ldots,$$

$$p_0 = b_0^{1/2}(x^2 - x_0^2) = Y + \varkappa_1 XY -$$

$$- \frac{1}{4}\frac{b_{02}'}{b_0^{3/2}} - \frac{k_1}{2} X^2 - \left(\frac{1}{6}k_{1S}' + \frac{1}{2}\varkappa_1 k_1\right) X^3 + . \tag{5}$$

From (5) we get the equation of the trajectory in the X, Y representation:

$$Y = aX^2 + bX^3, \qquad a = {}^1/_{10}(\ln J)_P',$$

$$b = {}^1/_{150}(4\varkappa_1 - \varkappa_2)(\ln J)_P' + {}^1/_{30} T_P'. \tag{6}$$

The curvature of the trajectory at O is thus dependent only on J:

$$k_t = {}^1/_5 (\ln J)_P'.$$

We put (6) in parametric form

$$X = X_e(u) = u, \qquad Y = Y_e(u) = au^2 + bu^3,$$

and construct a function that gives the image of the real axis in the plane $w = u + iv$ on the beam boundary in the plane of flow $Z = X + iY$:

$$Z = X + iY = w + i(aw^2 + bw^3),$$

$$X = u - k_t uv + b(v^3 - 3u^2 v),$$

$$Y = v + {}^1/_2 k_t (u^2 - v^2) + b(u^3 - 3uv^2). \tag{7}$$

The following is the approximate $Z \to w$ inverse image that coincides with the exact one up to cubic terms:

$$w = Z - {}^1/_2 i k_t Z^2 - ({}^1/_2 k_t^2 + ib) Z^3,$$

$$u = X + k_t XY + {}^1/_2 k_t^2 (3XY^2 - X^3) + b(3X^2 Y - Y^3),$$

$$v = Y - {}^1/_2 k_t (X^2 - Y^2) +$$

$$+ {}^1/_2 k_t^2 (Y^3 - 3X^2 Y) + b(3XY^2 - X^3). \tag{8}$$

Consider now the potential and its normal derivative on the trajectory. Since we intend to calculate k_φ (curvature of the zero-potential forming electrode at the origin) and k_φ (derivative of the previous), and since the main terms in the expansions of s_0, p_0, u, and v with respect to X and Y will be linear, it is sufficient to restrict ourselves to the following representation of the zero equipotential in the w plane:

$$v = \alpha u + \beta u^2 + \gamma u^3 \qquad (\alpha, \beta, \gamma = \mathrm{const}), \tag{9}$$

and in the expressions for the potential and the normal derivative at the boundary we substitute expressions that give terms of order $u^{4/3}$, $u^{7/3}$, and $u^{10/3}$ in the complex potential $W(u, v, w)$.

On the trajectory for s_0 and p_0 we have

$$s_0 = u - \frac{1}{4}\frac{a_1}{a_0^{3/2}} u^2 + \left[a k_1 + \frac{1}{6}\left(\frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2}\right) - \frac{1}{3}k_1^2\right] u^3,$$

$$p_0 = \left(a - \frac{k_1}{2}\right) u^2 + \left[b + a\varkappa_1 - \left(\frac{1}{6}k_{1S}' + \frac{1}{2}\varkappa_1 k_1\right)\right] u^3. \tag{10}$$

The potential in the space containing the charges is given by

$$2\varphi = \left(\frac{9J}{2}\right)^{2/3} s^{4/3}\left\{1 + \left(\frac{1}{3}\frac{a_1}{a_0^{3/2}} + \frac{8}{15}T\right)s + \right.$$
$$+\left[\frac{14}{45}\frac{a_1}{a_0^{3/2}}T - \frac{1}{24}\frac{a_1^2}{a_0^3} + \frac{2}{9}\frac{a_2}{a_0^2} + \right.$$
$$+\frac{83}{225}(\varkappa_1^2 + \varkappa_2^2) + \frac{157}{450}\varkappa_1\varkappa_2 - \frac{2}{9}k_1^2 +$$
$$\left.\left. + \frac{1}{3}k_1\frac{J_P'}{J} + \frac{4}{45}k_2\frac{J_P'}{J} + \frac{13}{450}\frac{J_P'^2}{J^2} - \frac{4}{45}\frac{J_P''}{J}\right]s^2\right\} \qquad (11)$$

The formulas of [3] may be simplified somewhat by using the conditions for the Euclidean nature of the space, as written in terms of the principal curvatures of the coordinate surfaces. These conditions take the following form for the axially symmetric case

$$\varkappa_{1S}' + k_{1P}' = \varkappa_1^2 + k_1^2, \quad \varkappa_{2S}' = \varkappa_2^2 + k_1k_2,$$
$$k_{2P}' = k_2^2 + \varkappa_1\varkappa_2,$$
$$\varkappa_{2P}' - k_2(\varkappa_2 - \varkappa_1) = 0, \quad k_{2S}' - \varkappa_2(k_2 - k_1) = 0. \qquad (12)$$

If the values at O are calculated in (3)–(8) and (10) for the curvature and for the coefficients in the expansions of the elements of the metric tensor, then all of these in (11) become functions of x^2. Then for the potential on the trajectory we have

$$2\varphi = V(u) = \left(\frac{9J}{2}\right)^{2/3} u^{4/3}\left\{1 + \frac{8}{15}Tu + \left[\frac{83}{225}(\varkappa_1^2 + \varkappa_2^2) + \right.\right.$$
$$\left.\left. + \frac{157}{450}\varkappa_1\varkappa_2 + \frac{43}{450}\frac{J_P'^2}{J^2} - \frac{4}{45}\frac{J_P''}{J} + \frac{4}{45}k_2\frac{J_P'}{J}\right]u^2\right\}. \qquad (13)$$

Note that V(u), as would be expected, is independent of the coordinate system used in solving the beam equations and is determined only by: 1) the geometry of the emitting surface; 2) the law of variation in J on that surface (k_2 is expressed in terms of $\varkappa_1$, $\varkappa_2$ and $\varkappa_2'_P$).

The following equation applies on the trajectory:

$$k_t(v_{x^1}^2 + v_{x^2}^2) = \partial\varphi / \partial n.$$

Up to quadratic terms, $k_t = 2a + 6bu$, and $v_{x1}^2 + v_{x2}^2 = 2\varphi$, by virtue of the existence of the energy integral. Finally:

$$2\left.\frac{\partial\varphi}{\partial v}\right|_{v=0} = F(u) =$$
$$= \left(\frac{9J}{2}\right)^{2/3} u^{4/3}\left\{\frac{2}{5}\frac{J_P'}{J} + \frac{2}{5}\left[T_P' + \frac{1}{3}\frac{J_P'}{J}(4\varkappa_1 + \varkappa_2)\right]u\right\}. \qquad (14)$$

The symbols A, B and C, D will be used in what follows for the coefficients in the braces in (13) and (14).

We now write the parametric equations for the trajectory in (z, R) coordinates and introduce the additional symbols

$$z = z_e(u) = z_0 + X_e(u)\cos\vartheta - Y_e(u)\sin\vartheta,$$
$$\beta(u) = -dz_e/du,$$
$$R = R_e(u) = R_0 + X_e(u)\sin\vartheta + Y_e(u)\cos\vartheta,$$
$$\alpha(u) = -dR_e/du. \qquad (15)$$

A solution has been given [5] to Cauchy's problem in the axially symmetric case, which was derived by transition to the complex region, in which the Laplace equation becomes hyperbolic, and Riemann's method is used:

$$2\varphi(u, v) = \operatorname{Re} W(u, v, w) = \operatorname{Re}\left\{\left[\frac{R_e(w)}{R}\right]^{1/2} V(w) + \right.$$
$$+ \frac{2}{\pi}\int_0^v \left[2R_e K(\sigma)F + 2R_e[K(\sigma) - \right.$$
$$\left. - E(\sigma)]V\frac{\alpha(z_e - z) - \beta(R_e - R)}{(R_e - R)^2 + (z_e - z)^2} - \beta E(\sigma)V\right] \times$$
$$\left.\times \frac{d\xi}{[(R_e + R)^2 + (z_e - z)^2]^{1/2}}\right\},$$
$$\sigma = \left[\frac{(R_e - R)^2 + (z_e - z)^2}{(R_e + R)^2 + (z_e - z)^2}\right]^{1/2}. \qquad (16)$$

Here $K(\sigma)$ and $E(\sigma)$ are complete elliptic integrals of the first and second kinds, while R_e, z_e, V, F, α, and β are abbreviations for $R_e(u + i\xi)$, $z_e(u + i\xi)$, $V(u + i\xi)$, $F(u + i\xi)$, $\alpha(u + i\xi)$, $\beta(u + i\xi)$, respectively:

$$z = z_0 + [\operatorname{Re} X_e(w) - \operatorname{Im} Y_e(w)]\cos\vartheta -$$
$$- [\operatorname{Im} X_e(w) + \operatorname{Re} Y_e(w)]\sin\vartheta$$
$$R = R_0 + [\operatorname{Im} X_e(w) + \operatorname{Re} Y_e(w)]\cos\vartheta +$$
$$+ [\operatorname{Re} X_e(w) - \operatorname{Im} Y_e(w)]\sin\vartheta \qquad (17)$$

In considering the first term in (16), it must be borne in mind that R corresponds to a point close to O, while terms linear in u and ξ are to be retained in the coefficients of V and F in evaluating the integral. Here we use expansions [6] for the complete elliptic integrals for small values of the argument

$$K(\sigma) = {}^1/_2\,\pi(1 + {}^1/_4\,\sigma^2 + \ldots),$$
$$E(\sigma) = {}^1/_2\,\pi(1 - {}^1/_4\,\sigma^2 + \ldots),$$
$$K(\sigma) - E(\sigma) = {}^1/_2\,\pi\cdot{}^1/_2\,\sigma^2.$$

We then get the following expression for the solution to Laplace's equation near O :

$$2\varphi(u, v) = \left[1 - \frac{\sin\vartheta}{2R_0}u - \frac{\cos\vartheta}{2R_0}v + \right.$$
$$+\left(\frac{3\sin 2\vartheta}{8R_0^2} + \frac{a\sin\vartheta}{R_0}\right)uv + \left(\frac{3\sin^2\vartheta}{8R_0^2} - \frac{a\cos\vartheta}{2R_0}\right)u^2 +$$
$$\left. + \left(\frac{3\cos^2\vartheta}{8R_0^2} + \frac{a\cos\vartheta}{2R_0}\right)v^2\right](u^2 + v^2)^{2/3} \times$$
$$\times \cos\frac{4}{3}\operatorname{arc\,tg}\frac{v}{u} + \left[A + \frac{\sin\vartheta}{2R_0} - \right.$$
$$- \left(\frac{A\sin\vartheta}{2R_0} + \frac{\sin^2\vartheta}{4R_0^2}\right)u -$$
$$\left. - \left(\frac{A\cos\vartheta}{2R_0} + \frac{\sin 2\vartheta}{8R_0^2}\right)v\right](u^2 + v^2)^{7/6}\cos\frac{7}{3}\operatorname{arc\,tg}\frac{v}{u} +$$
$$+ \left(B + \frac{A\sin\vartheta}{2R_0} + \frac{a\cos\vartheta}{2R_0} - \right.$$
$$\left. - \frac{\sin^2\vartheta}{8R_0^2}\right)(u^2 + v^2)^{5/3}\cos\frac{10}{3}\operatorname{arc\,tg}\frac{v}{u} +$$
$$+ \frac{3}{7}\left[C + \frac{\cos\vartheta}{2R_0} - \left(\frac{\sin 2\vartheta}{8R_0^2} + \frac{C\sin\vartheta}{2R_0}\right)u - \right.$$
$$\left. - \left(\frac{1}{4R_0^2} + \frac{\cos 2\vartheta}{8R_0^2} + \frac{C\cos\vartheta}{2R_0}\right)v\right](u^2 +$$
$$+ v^2)^{7/6}\sin\frac{7}{3}\operatorname{arc\,tg}\frac{v}{u} + \frac{3}{10}\left(D + \frac{A\cos\vartheta}{2R_0} - \frac{a\sin\vartheta}{R_0} + \right.$$
$$\left. + \frac{C\sin\vartheta}{2R_0} - \frac{\sin 2\vartheta}{8R_0^2}\right)(u^2 + v^2)^{5/3}\sin\frac{10}{3}\operatorname{arc\,tg}\frac{v}{u}. \qquad (18)$$

The following are the coefficients that define the explicit equation for the zero equipotential in the w plane:

$$\alpha = \operatorname{tg}\frac{3\pi}{8}, \quad \beta = \frac{3}{4}(1 + \alpha^2)^{3/2} \times$$
$$\times \left[-\left(A + \frac{\sin\vartheta}{2R_0}\right)\cos\frac{\pi}{8} + \frac{3}{7}\left(C + \frac{\cos\vartheta}{2R_0}\right)\sin\frac{\pi}{8}\right],$$
$$\gamma = \frac{2\alpha\beta^2}{1 + \alpha^2} + \frac{\beta}{2R_0}\cos\left(\vartheta - \frac{\pi}{8}\right) -$$
$$- \frac{7}{4}(1 + \alpha^2)\beta\sin\frac{\pi}{8}\left[\left(A + \frac{\sin\vartheta}{2R_0}\right)\sin\frac{\pi}{8} + \right.$$
$$\left. + \frac{3}{7}\left(C - \frac{\cos\vartheta}{2R_0}\right)\cos\frac{\pi}{8}\right] +$$

$$+\frac{3}{8R_0}(1+\alpha^2)\,\alpha\left(A+\frac{\sin\vartheta}{2R_0}\right)\cos\left(\vartheta-\frac{\pi}{8}\right)-$$
$$-\frac{3}{4\sqrt{2}}\left[B+\frac{A\sin\vartheta}{2R_0}+\frac{a\cos\vartheta}{2R_0}-\right.$$
$$-\frac{\sin^2\vartheta}{8R_0^2}+\frac{3}{10}\left(D-\frac{A\cos\vartheta}{2R_0}+\frac{a\sin\vartheta}{R_0}+\right.$$
$$\left.\left.+\frac{C\sin\vartheta}{2R_0}+\frac{\sin 2\vartheta}{8R_0^2}\right)\right]+\frac{2}{7R_0}(1+\alpha^2)\left[\frac{\alpha}{2R_0}-\right.$$
$$\left.-C\cos\left(\vartheta-\frac{\pi}{8}\right)+\frac{1}{4R_0}\cos\left(2\vartheta-\frac{\pi}{8}\right)\right].$$

To put the equation for $\varphi = 0$ in local Cartesian coordinates

$$Y=\mu X+\nu X^2+\lambda X^3\,, \tag{19}$$

we use (18). The curvature of the zero-potential forming electrode and k'_φ are given by

$$k_\varphi=2\nu\,(1+\mu^2)^{-3/2},\qquad k_\varphi{}'=6\,[(1+\mu^2)\,\lambda-2\mu\nu^2](1+\mu^2)^{-5/2},$$
$$\mu=\alpha,\qquad \nu={}^1/_2\,(1+\alpha^2)\,k_t+\beta,$$
$$\lambda=\alpha\,(1+\alpha^2)\,k_t^2+b\,(1-\alpha^4)+2\alpha\beta k_t+\gamma.$$

Omitting intermediate steps, we have

$$k_\varphi=-\left(\frac{4}{5}T+\frac{3}{4}\frac{\sin\vartheta}{R_0}\right)\cos\frac{\pi}{8}+$$
$$+\left(\frac{16}{35}\frac{J'_P}{J}+\frac{9}{28}\frac{\cos\vartheta}{R_0}\right)\sin\frac{\pi}{8}\,,$$
$$k_\varphi{}'=6\cos\frac{\pi}{8}\left[-\frac{37}{150}T_P{}'-\frac{9}{50}(\varkappa_1^2+\varkappa_2^2)+\right.$$
$$+\frac{67}{300}\varkappa_1\varkappa_2-\frac{4}{75}\varkappa_1\frac{J_P{}'}{J}+$$
$$+\frac{29}{150}\varkappa_2\frac{J_P{}'}{J}-\frac{2}{15}k_2\frac{J_P{}'}{J}+\frac{2}{15}\frac{J_P{}''}{J}-$$
$$-\frac{17}{105}\frac{J_P{}'^2}{J^2}+\left(\frac{3}{10}T+\frac{9}{50}\frac{J_P{}'}{J}\right)\frac{\sin\vartheta}{R_0}+$$
$$+\left(\frac{9}{50}T+\frac{3}{140}\frac{J_P{}'}{J}\right)\frac{\cos\vartheta}{R_0}-$$
$$-\frac{1}{R_0^2}\left(\frac{1}{7}-\frac{9}{64}\sin^2\vartheta+\frac{27}{448}\cos^2\vartheta-\right.$$
$$\left.\left.-\frac{63}{320}\sin 2\vartheta\right)\right]-6\sin\frac{\pi}{8}\left[\frac{1}{14R_0^2}\cos\left(2\vartheta-\frac{\pi}{8}\right)+\right.$$
$$\left.+\frac{1}{R_0}\left(\frac{1}{20}\frac{J_P{}'}{J}+\frac{9}{112}\frac{\cos\vartheta}{R_0}\right)\cos\left(\vartheta-\frac{\pi}{8}\right)\right]. \tag{20}$$

Formulas (20) can be entirely recast in terms of the curvature of the emitting surface if we note that

$$k_2=\varkappa'_{2P}\,(\varkappa_2-\varkappa_1)^{-1},\qquad R_0^{-2}=\varkappa_2^2+k_2^2,\qquad \sin\vartheta=-\varkappa_2R_0\,.$$

We allow R_0 to tend to infinity and put $\varkappa_1 = \varkappa$, $\varkappa_2 = k_2 = 0$ in (20) to get for the planar case that

$$k_\varphi=-\frac{4}{5}\varkappa\cos\frac{\pi}{8}+\frac{16}{35}\frac{J_P{}'}{J}\sin\frac{\pi}{8}\,,$$
$$k_\varphi{}'=6\cos\frac{\pi}{8}\left(-\frac{37}{150}\varkappa_P{}'-\frac{9}{50}\varkappa^2-\right.$$
$$\left.-\frac{4}{75}\varkappa\frac{J_P{}'}{J}+\frac{2}{15}\frac{J_P{}''}{J}-\frac{17}{105}\frac{J_P{}'^2}{J^2}\right). \tag{21}$$

To conclude, we consider x^1 flows [7, 8], for which the equation for the trajectory is $p_0 = 0$ or

$$Y=aX^2+bX^3,\qquad a={}^1/_2k_1,$$
$$b={}^1/_6k_{1S}{}',\qquad k_t=k_1={}^1/_5\,(\ln J)_P{}'. \tag{22}$$

These are allowed in a plane by coordinate systems having the metric

$$g_{11}=g_{22}=\sqrt{g}=\exp\,(\varepsilon x^1+\tau x^2)\qquad (\varepsilon,\ \tau=\text{const})\,, \tag{23}$$

while in three dimensions they are allowed by spherical coordinates. It is clear [3] that $v_2 \equiv 0$ if $U_{02}' = 0$, and U_k/U_0 = constant. It is readily seen that the second condition is obeyed for (23), while the first gives the law of variation of J at $x^1 = 0$:

$$J=J_0\exp\,(-{}^5/_2\tau x^2).$$

The expressions for k_φ are unaltered. From (23)

$$k_\varphi={}^2/_5\;\varepsilon\cos\pi/8\mp{}^8/_7\;\tau\sin\pi/8\,. \tag{24}$$

The minus sign applies when the Laplace region extends towards increasing x^2, while the plus sign applies in the converse case. For instance, for a wedge-shaped beam (part of a cylindrical diode) and for flow around circles we have

$$g_{11}=g_{22}=\exp\,(2x^1),\quad x^1=\ln R,\quad x^2=\psi,$$
$$\varepsilon=2,\quad \tau=0,\quad J=\text{const},\quad k_\varphi={}^4/_5\cos\pi/8\,,$$
$$g_{11}=g_{22}=\exp\,(2x^2),\quad x^1=\psi,\quad x^2=\ln R,\quad \varepsilon=0,$$
$$\tau=2,\quad J=J_0R^{-5},\quad k_\varphi=\mp{}^{16}/_7\sin\pi/8\,.$$

In view of (22), we have as follows for k'_φ in spherical coordinates and in systems (23):

$$k_\varphi{}'=6\cos\frac{\pi}{8}\left[-\frac{9}{50}T_P{}'-\frac{9}{50}(\varkappa_1^2+\varkappa_2^2)+\right.$$
$$+\frac{67}{300}\varkappa_1\varkappa_2+\frac{9}{50}\varkappa_2\frac{J_P{}'}{J}-$$
$$-\frac{2}{15}k_2\frac{J_P{}'}{J}+\frac{2}{15}\frac{J_P{}''}{J}-\frac{17}{105}\frac{J_P{}'^2}{J^2}-\frac{1}{6}k_{1S}{}'+$$
$$+\left(\frac{3}{10}T+\frac{9}{50}\frac{J_P{}'}{J}\right)\frac{\sin\vartheta}{R_0}+$$
$$+\left(\frac{9}{50}T+\frac{3}{140}\frac{J_P{}'}{J}\right)\frac{\cos\vartheta}{R_0}-$$
$$-\frac{1}{R_0^2}\left(\frac{1}{7}-\frac{9}{64}\sin^2\vartheta+\frac{27}{448}\cos^2\vartheta-\right.$$
$$\left.\left.-\frac{63}{320}\sin 2\vartheta\right)\right]-6\sin\frac{\pi}{8}\left[\frac{1}{14R_0^2}\cos\left(2\vartheta-\frac{\pi}{8}\right)+\right.$$
$$\left.+\frac{1}{R_0}\left(\frac{1}{20}\frac{J_P{}'}{J}+\frac{9}{112}\frac{\cos\vartheta}{R_0}\right)\cos\left(\vartheta-\frac{\pi}{8}\right)\right],$$
$$k_\varphi{}'=6\cos\frac{\pi}{8}\left(-\frac{9}{50}\varkappa_P{}'-\frac{9}{50}\varkappa^2+\right.$$
$$\left.+\frac{2}{15}\frac{J_P{}''}{J}-\frac{17}{105}\frac{J_P{}'^2}{J^2}-\frac{1}{6}k_{1S}{}'\right).$$

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REFERENCES

1. A. R. Lucas, "The relativistic flow of electrons in parallel and radial straight lines with no externally imposed magnetic field," J. Electr. Contr., vol. 5, no. 3, 1958.
2. D. E. Radley, "The theory of the pierce type electron gun," J. Electr. Contr., vol. 4, no. 2, 1958.
3. Yu. E. Kuznetsov and V. A. Syrovoi, "Solution of the equations for a regular electrostatic beam with emission from an arbitrary surface," PMTF [Journal of Applied Mechanics and Technical Physics], no. 2, 1966.
4. V. A. Syrovoi, "Solution of the equations for a regular beam with arbitrary emission conditions at a curvilinear surface," PMTF [Journal of Applied Mechanics and Technical Physics], no. 3, 1966.

5. K. J. Harker, "Solution of the Cauchy Problem for Laplace's Equation in axially symmetric systems," J. Math. Phys., vol. 4, no. 7, 1963.

6. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Fizmatgiz, 1962.

7. V. A. Syrovoi, "One-component beams of similarly charged particles," PMTF, no. 3, 1964.

8. V. A. Syrovoi, "Theory of regular electrostatic beams of charged particles," PMTF [Journal of Applied Mechanics and Technical Physics], no. 1, 1966.

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